Asymptotically Stable Solution of a System of Nonlinear Multi-Fractional Order Differential Equations

By
A.N. Ahmed and R.A. Abdul-Satar

Abstract

In this paper, a theorem for the stability of a system of nonlinear multi-fractional order differential equations on an infinite time interval is presented, by using the fractional difference method to construct an approximating solution based on the definition of the fractional derivative of the Grunwall-Letnikov.

الخلاصة

في هذا البحث تم عرض نظرية حول أستقرارية الحل لنظام من المعادلات التفاضلية الغير خطية ذات الرتب الكسرية المتعددة لفترة غير منتهية واستخدام طريقة الفروقات الكسرية للحصول على حلول تقريبية باعتماد تعريف كرانويل – لينكتون للمشتقة ذات الرتب الكسرية.

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Introduction

Few lecturers had been published on the stability of fractional order of nonlinear system , such as [1],[5] , [6],[7] &[8].

Fractional derivatives have a long mathematical history, for many years they were not used in many different sciences, but in recent years, growing attention has been focused on the importance of fractional derivatives and integrals in science. (or more details, see [4]).

Problems of stability appeared for the first time in mechanics during the investigation of an equilibrium state of a system. A simple reflection may show that some equilibrium state of a system are stable with respect to small perturbations. The existing methods developed so far for studying the stability are mainly for integer order systems. However, for the fractional order systems, it is difficult to evaluate the stability by simply examining its characteristic equation either by finding its dominate roots or by using other algebraic methods. Direct study of the stability of fractional order system using polynomial criteria is not possible, because the characteristic equation of the system is, in general, not a polynomial but a pseudopolynomial function of fractional powers of the complex variables.(for more details , see [8]).

The study of stability of such systems focuses a great interest. We can cite in this domain, the works in [7] and [8] for the stability of linear fractional systems, while the works in [5], and [6] are for the stability of fractional systems with time delay.

In [1], Ahmed and etal. are concerned with the stability of nonlinear multi-fractional order system of differential equations

(1) …. $y_i^{(\alpha_i)}(t) = f_i(t, Y(t))$ i=1,2, … ,m. $m \in N$

Where $Y(t) = (y_1(t), \ldots, y_m(t))^T$ is its solution, 0<, and $f_i \in C(R^+ \times R^n, R^{n+})$ are continuous positive functions defined on a finite time interval with the prescribed given condition.

In this paper, we will extend the work of [1] for studying the stability of the system (1) on an infinite time interval and $0 < \alpha_i < 1$, by modifying the definition of fractional derivative, making use of some properties of fractional integrals.

Preliminaries

Several definitions of a fractional derivative of order α i>0 are there (see [4]). The two most commonly used definitions are those due by Riemann-Liouville and Caputo. Each definition uses Riemann-Liouville fractional integration and derivatives of whole order. In this paper, we are using the Riemann-Liouville fractional integration of order αi which is defined as follows :

$$y_i(t) = I^{\alpha_i} f_i(t, Y(t)) = \frac{y_0 (t-t_0)^{\alpha_i - 1}}{\Gamma(\alpha_i)} + \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} f(s, Y(s)) ds,$$

(2) …

$$= \frac{t^{\alpha_i - 1}}{\Gamma(\alpha_i)} \otimes f(t, Y(t)), \qquad \alpha_i > 0,\ t > 0.$$

where $\otimes$ is the convolution operator.

The next two equations define Riemann-Liouville and Caputo fractional derivatives of order α, respectively,

$$D^{\alpha} f(t) = \frac{d^{\alpha}}{dt^{\alpha}} (I^{n-\alpha} f(t)) \qquad \text{…(3)}$$

$$D^{\alpha} f(t) = I^{n-\alpha} (\frac{d^{\alpha}}{dt^{\alpha}} (f(t)) \qquad \text{…(4)}$$

where, $n-1 < \alpha < n$ and n=[α]+1.

where [α] is the greatest integer less than α.

A very useful fact about the R_L operators is that they satisfy the following important properties of fractional integrals (see [4])

For any $f \in C([a,b]),\ \alpha > 0, and\ \beta > 0$ the R_L fractional integral satisfies

$$I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t) \qquad \text{…(5)}$$

$$D^{\alpha} I^{\alpha} f(t) = f(t) \qquad \text{…(6)}$$

$I^{\alpha} D^{\alpha} f(t) = f(t)$, if f satisfies that $(\frac{d^{k}}{dt^{k}} I^{n-\alpha} f(t))_{|t=a} = 0$, k=0,1,2, … ,n-1. …(7)

The main result

In [1], we investigated the L^p- stability properties of a system of nonlinear multi-fractional order differential equation. We are presented a theorem, which gives the sufficient conditions for bounded stability of such system on a finite time interval, by defining the solution of the system using the convolution operator. We show that the proposed results can not be extended to the case of systems defined on an infinite time interval, because the kernel function in the solution does not belong to $L^1(R^+)$, and the solution of the system can not be defined by using the convolution product on an infinite case.

In this study, in order to overcome the difficulties encountered in [1], we are considering the definition of the fractional derivative in the Grunwald-Letnikov sense which is defined as

$$D^{\alpha} y(t) = \lim_{h\to 0}(h)^{-\alpha} \sum_{j=0}^{[\frac{t}{h}]} (-1)^j \binom{\alpha}{j} y(t-jh) \qquad \ldots(8)$$

where [t] is the integer part of t, and h is the step size. The definition of the operator in the Grunwald Letnikov
sense (8) is equivalent to the definition of operator in the Riemann-Liouville sense (2) and(3). Nevertheless, the Grunwald-Letnikov operator is more flexible and most straightforward in the numerical calculations, in which, the solution of (8) is given by the following recurrence equation: see [9]:

$$y_o = \beta,\ y_n = f(t, y_n) - \sum_{j=1}^{n} c_j^{\alpha} y_{n-j}, \quad (n=1,2,3,\ldots) \qquad \ldots(9)$$

Where

$$t_n = nh,\ y_n = y(t_n) \qquad \ldots(10)$$

Which are positive functions for all n, c_j^{α} are the Grunwald-Letnikov coefficients defined as:

$$c_0^{\alpha} = h^{-\alpha}, \quad c_j^{\alpha} = \left(1 - \frac{1+\alpha}{j}\right) c_{j-1}^{\alpha}, \quad (j=1,2,3,\ldots) \qquad \ldots(11)$$

For more details, see [7].

If we denote $y_i(t) = \lim_{n\to\infty} y_n$, for all i=1,…,m, and $f(t, Y(t)) = \lim_{n\to\infty} f(t, Y(t_n))$. Define the norm of Y(t) as follows

$${}_0^\infty\| Y(t) \| = \sum_{i=1}^{m} {}_0^\infty\| y_i(t) \|, \qquad \ldots(12)$$

In order to study the stability of the solution of the system (1), we must prove that each $y_i(t)$, (i=1,…,m) is bounded, i.e. $\| y_i(t) \| < \infty$, for all i=1,…,m.

Now, we can state and prove the following theorem:

Theorem:

The solution of (1) is asymptotically stable, providing that $f_i \in L^p[0,\infty)$, with p≥ 1, (for all i=1,…,m).

Proof: Consider the solution (9).

First, we will prove that the system (1) is stable.

$$\| y_i(t) \| = \| f_i(t, Y(t)) - \sum_{j=1}^{n} c_j^{\alpha_i} y_i(t_{n-j}) \|$$

From (11), $0 < c_j^{\alpha_i} < 1,$ for all j=1,2,3,…,n, and all $0 \leq y_i(t_{n-j}) \leq \beta$ then all $\lim_{n\to\infty} \sum_{j=1}^{n} c_j^{\alpha_i} y_i(t_{n-j})$ is converge and bounded, and since $f_i \in L^p(0,\infty)$, with p≥ 1, (for all i=1,…,m), the right hand side is bounded, and we have

$\| y_i(t) \| < \infty$, and then $\| Y(t) \| < \infty$.

Second, if we consider two solutions Y(t) and X(t) with two different initial values $Y_o \, and \, X_o$, with

$| y_i(t_o) - x_i(t_o) | < \epsilon$, for all i=1,…,m, and 0<ϵ<1, then $\lim_{n\to\infty} \left(y_i(t_{n-j}) - x_i(t_{n-j}) \right) \to 0$ and $\lim_{n\to\infty} (f_i(t, Y(t_n)) - f_i(t, X(t_n))) \to 0 \;\; as \; n \to \infty$. Performing same as above steps, we get $\| Y(t) - X(t) \| \to 0, \; as \, t \to \infty$. Therefore, the system (1) is asymptotically stable.

Conclusion

As we have seen, the difficulties in [1] may be solved by using the fractional difference method to construct an approximating solution based on the definition of the fractional derivative of the Grunwall-Letnikov, to prove the stability of the system (1), in case of infinite time interval with less conditions.

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